Journal of Statistical Physics, Vol. 77, Nos. 3/4, 1994

Numerical Solution of Differential Equations with Colored Noise

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Received October 5, 1993; final May 2, 1994

Using the general theory of numerical integration of stochastic differential equations, a constructive approach to numerical methods for a system with colored noise is proposed. Efficient methods up to the 5/2 strong order and up to the third weak order, including Runge–Kutta and implicit schemes, are presented. The algorithms are tested on the Kubo oscillator.

KEY WORDS: Stochastic differential equations; colored noise; numerical methods.

1. INTRODUCTION

Stochastic differential equations (SDE) have found applications in many fields of research,[1-5] including chemical physics, laser noise problems, combustion, mathematical biology, etc. The simplest approximation of real fluctuations that affect a physical system is Gaussian white noise. However, Gaussian white noise, or a Gaussian delta-correlated random process, is a stochastic process with zero correlation time and infinite variance, so it is an unreal process which has no evident physical sense. Such a random process may be considered only as the first approximation of real fluctuations with a short correlation time.[6,7] This shortcoming is overcome by colored noise (finite-bandwidth noise).[1,6] Recently several authors have studied exponentially correlated colored noise. The investigation of systems with colored noise was stimulated to a large extent by the occurrence of correlated pump noise in dye lasers.[3] Analytical results have been obtained for weakly[8] and highly (strongly)[9] colored noise. Appearance of nonlinear and complex stochastic differential equations in theoretical

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0022-4715/94/1100-0691$07.00/0

models has led to the necessity for numerical simulation of such SDE. Different algorithms for differential equations with colored noise have been proposed.[10-13]

Herein we apply common methods of the theory of numerical integration of SDE[14,15] to differential equations with exponentially correlated colored noise

$$\begin{aligned} dY &= f(Y)\,dt + G(Y)Z\,dt \\ dZ &= AZ\,dt + \sum_{r=1}^{q} b_r\,dW_r \end{aligned} \tag{1.1}$$

where Y and f are one-dimensional vectors, Z and b_r are m-dimensional vectors, A is an $m \times m$ matrix, G is an $l \times m$ matrix, and W_r are uncorrelated standard Wiener processes. In the one-dimensional case Eqs. (1.1) are rewritten in the form

$$\begin{aligned} dy &= f(y)\,dt + g(y)z\,dt \\ dz &= -az\,dt + b\,dW \end{aligned} \tag{1.2}$$

where z is the well-known Ornstein–Uhlenbeck process, or exponentially correlated colored noise, with the properties

$$\langle z(t) \rangle = 0, \qquad \langle z(t)\,z(s) \rangle = \frac{b^2}{2a} \exp(-a\,|t-s|) \tag{1.3}$$

The system (1.1) is simpler than the general one [see (2.1)] by two reasons: (1) (1.1) is a system with additive Gaussian white noises, (2) Eqs. (1.1) are linear with respect to Z. That is why comparatively simple high-order methods may be constructed for numerical solution of differential equations with colored noise.

In the earlier works[10-13] authors obtained efficient (as to simulation of the used random variables) explicit algorithms up to the second order in the strong sense for the numerical integration of colored noise problems.

Here for the first time on the basis of the general theory various methods for the system (1.1) are easily obtained. We create strong explicit methods up to the 5/2 order in which random variables are simulated in a simple way. Let us note that in the case of a general stochastic system [see (2.1)] the first-order strong schemes already require calculation of repeated Ito integrals, which is a difficult and laborious problem,[14] and the resulting algorithms become overly unwieldy and inefficient. Fortunately, thanks to the special features of the system (1.1), we succeeded in constructing efficient high-order methods for the numerical solution of differential equations with colored noise. Moreover, we present efficient implicit and Runge–Kutta schemes up to the second order. We give particular attention to efficient weak approximations of the solution of system (1.1) and obtain weak explicit schemes up to the third order, weak implicit methods up to the second order, and explicit and implicit Runge–Kutta schemes.

The paper is organized as follows. In Section 2 some needed facts of the general theory of numerical integration of SDE are formulated. They are useful for derivation of the numerical methods of Sections 3 and 4. This section is based on refs. 14 and 15, which contain the results of the well-developed modern theory of numerical integration of SDE. In contrast to the deterministic case, for stochastic equations we can consider various types of approximations; the most common and useful in practice are mean-square (strong) and weak approximations. The first investigations of strong approximations were reported in refs. 16 and 17, while refs. 18–20 are the first studies on weak approximations. In Section 3 strong algorithms for a system with colored noise are presented. Section 4 is devoted to weak schemes. In Section 5 numerical tests of the presented methods are discussed. We restrict ourselves to the full proof of only one strong method, namely the trapezoidal scheme (see Appendix). Other methods may be proved in a similar way.

2. SOME NEEDED FACTS OF THE GENERAL THEORY OF SDE NUMERICAL INTEGRATION

In this section we shall be concerned with the common system of the Ito equations

$$
\begin{aligned}
dX^i &= a^i(X)\,dt + \sum_{r=1}^{q} \sigma_r^i(X)\,dW_r(t) \\
X^i(t_0) &= X_0^i, \qquad t \in [t_0, T], \qquad i = 1, \ldots, n
\end{aligned}
\tag{2.1}
$$

Note that the initial value $X_0 = X(t_0)$ may have either a sharp (deterministic) value or it may already be a stochastic variable, the probability of which follows from the initial distribution. For instance, the initial value z_0 of the Ornstein–Uhlenbeck process (1.2)–(1.3) is distributed as a Gaussian variable with zero mean and variance $b^2/(2a)$.

Let a discretization of the interval $[t_0, T]$ be defined as $S_N = \{t_i: 0, 1, \ldots, N;\ t_0 < t_1 < \cdots < t_N = T\}$. Below we use the following notation: the time increment $h = t_{i+1} - t_i$, the approximation X_k or $\bar{X}(t_k)$ of the exact solution $X(t_k)$, operators

$$L = \left(a, \frac{\partial}{\partial X}\right) + \frac{1}{2} \sum_{r=1}^{q} \left(\sigma_r, \frac{\partial}{\partial X}\right)^2$$

$$= \sum_{i=1}^{n} a^i \frac{\partial}{\partial X^i} + \frac{1}{2} \sum_{r=1}^{q} \sum_{i,j=1}^{n} \sigma_r^i \sigma_r^j \frac{\partial^2}{\partial X^i \partial X^j}$$

$$\Lambda_r = \left(\sigma_r, \frac{\partial}{\partial X}\right) = \sum_{i=1}^{n} \sigma_r^i \frac{\partial}{\partial X^i}$$

and Ito integrals

$$I_{i_1, i_2, \dots, i_{j_k}}(h) = \int_{t_k}^{t_k+h} dW_{i_j}(\theta) \int_{t_k}^{\theta} dW_{i_{j-1}}(\theta_1) \int_{t_k}^{\theta_1} \cdots \int_{t_k}^{\theta_{j-2}} dW_{i_1}(\theta_{j-1})$$

where $i_1, \dots, i_j$ are from the set of numbers $\{0, 1, \dots, q\}$ and $dW_0(\theta_r)$ is equal to $d\theta_r$.

2.1. Strong Approximation

Definition 1. If the inequality

$$[\langle |X(t_k) - X_k|^2 \rangle]^{1/2} \leqslant Ch^p \tag{2.2}$$

holds for a numerical method, where C does not depend on k and h, and p is greater than zero, the mean-square order of the method is equal to p.

It must be mentioned that C in (2.2) depends on initial values X_0.

An important practical application of the strong approximations is the direct simulation of stochastic dynamical systems. Direct simulations of trajectories of SDE can provide useful information on the qualitative behavior of a model. Another practical application of mean-square approximations is to the problem of estimating parameters. Note that strong methods are also the basis for the construction of weak ones.

A theorem on the relation between the order of one-step strong approximation and the mean-square order of the corresponding method on the whole interval has been proposed in refs. 14 and 21 and here it is only stated:

Theorem 1. Let us assume that the one-step approximation $\bar{X}(t+h)$ satisfies the inequalities

$$|\langle X(t+h) - \bar{X}(t+h) \rangle| \leqslant K(1+|x|^2)^{1/2} h^{p_1}$$

$$[\langle |X(t+h) - \bar{X}(t+h)|^2 \rangle]^{1/2} \leqslant K(1+|x|^2)^{1/2} h^{p_2}$$

where

$$\bar{X}(t) = X(t) = x, \qquad t \in [t_0, T-h], \qquad x \in \mathbf{R}^n, \qquad p_2 \geqslant \tfrac{1}{2}, \qquad p_1 \geqslant p_2 + \tfrac{1}{2}$$

Then

$$[\langle |X(t_k) - \bar{X}(t_k)|^2 \rangle]^{1/2} \leqslant K(1 + \langle |X_0|^2 \rangle)^{1/2} h^{p_2 - 1/2}$$

for any N and $k = 0, 1, ..., N$, i.e., the mean-square order of the method, based on the one-step approximation $\bar{X}(t+h)$, is equal to $p = p_2 - \frac{1}{2}$.

In refs. 14, 15, and 17 a formula of the Taylor type for expanding the solution of a stochastic differential equation about the points of a time partition has been obtained. The formula is named the Wagner–Platen expansion. Let us describe the rule of construction of the Wagner–Platen expansion. According to the Ito formula, a sufficiently smooth function $f(x)$ is written as

$$f(X(\theta)) = f(X(t)) + \sum_{r=1}^{q} \int_t^\theta \Lambda_r f(X(\theta_1))\, dW_r(\theta_1) + \int_t^\theta Lf(X(\theta_1))\, d\theta_1 \qquad (2.3)$$

where $X(t)$ is a solution of the system (2.1), $t_0 \leqslant t \leqslant \theta \leqslant T$. The second and the third terms of (2.3) also may be represented by the Ito formula [for instance,

$$\Lambda_r f(X(\theta_1) = \Lambda_r f(X(t)) + \sum_{s=1}^{q} \int_t^{\theta_1} \Lambda_s \Lambda_r f(X(\theta_2))\, dW_s(\theta_2) + \int_t^{\theta_1} L\Lambda_r f(X(\theta_2))\, d\theta_2$$

and therefore

$$\int_t^\theta \Lambda_r f\, dW_r(\theta_1) = \Lambda_r f(X(t)) \int_t^\theta dW_r + \int_t^\theta \sum_{s=1}^{q} \int_t^{\theta_1} \Lambda_s \Lambda_r f(X(\theta_2))\, dW_s(\theta_2)\, dW_r(\theta_1) + \int_t^\theta \int_t^{\theta_1} L\Lambda_r f(X(\theta_2))\, d\theta_2\, dW_r(\theta_1)\Bigg]$$

then substituted in the expression (2.3) and so on. Thus, we have obtained the Wagner–Platen expansion,[14,15,17] which is similar to the Taylor expansion in the deterministic case. If function $f(x)$ is equal to x, then $Lf \equiv a$, $\Lambda_r f \equiv \sigma_r$, and

$$X(t+h) = X(t) + \sum_{r=1}^{q} \sigma_r I_r + ah + \sum_{r,i=1}^{q} \Lambda_i \sigma_r I_{ir}$$
$$+ \sum_{r=1}^{q} L\sigma_r I_{0r} + \sum_{r=1}^{q} \Lambda_r a I_{r0}$$
$$+ \sum_{r=1}^{q} \sum_{i=1}^{q} \sum_{s=1}^{q} \Lambda_s \Lambda_i \sigma_r I_{sir} + La\frac{h^2}{2} + \rho \qquad (2.4)$$

where the remainder ρ is easily calculated by the described rule.

2.2. Weak Approximation

Definition 2. $\bar{X}(t)$ approximates the exact solution $X(t)$ in the weak sense with order p if

$$|\langle f(\bar{X}(t))\rangle - \langle f(X(t))\rangle| = O(h^p), \qquad t \in [t_0, T] \qquad (2.5)$$

for any sufficiently smooth function f.

The weak approximations are of great importance; first, they are sufficient for most physical problems, and second, they are simpler and more constructive than the strong ones. Weak numerical methods can be used for calculations of statistics of stochastic processes (for instance, mean value, variance, mean first passage time, etc), Wiener functional space integrals, for solving problems of mathematical physics by Monte Carlo technique, etc. Weak approximations use random variables, which are easily simulated, and include much fewer terms with operators than do strong methods.

Similar to Theorem 1, a theorem on the relation between the properties of the one-step weak approximation and the weak order of a method on the whole interval has been obtained.[14,15,19,20] According to this theorem, the weak order of the method is equal to p if

$$\left|\left\langle \prod_{j=1}^{s} \bar{\Delta}^{i_j} - \prod_{j=1}^{s} \Delta^{i_j} \right\rangle\right| \leqslant Ch^{p+1}$$

where $\Delta = X(t+h) - x$, $\bar{\Delta} = \bar{X}(t+h) - x$, $X(t) = \bar{X}(t) = x$, and $s = 1, \ldots, 2p+1$. In refs. 14, 15, 19, and 20 one can find the full formulation of the theorem.

3. STRONG METHODS FOR DIFFERENTIAL EQUATIONS WITH COLORED NOISE

For the system (1.1) operators the L and Λ_r take the form

$$L=\left(f(Y)+G(Y)Z,\frac{\partial}{\partial Y}\right)+\left(AZ,\frac{\partial}{\partial Z}\right)+\frac{1}{2}\sum_{r=1}^{q}\sum_{i,j=1}^{m}b_r^i b_r^j\frac{\partial^2}{\partial z^i\,\partial z^j}$$
$$\Lambda_r=\left(b_r,\frac{\partial}{\partial Z}\right) \tag{3.1}$$

In the one-dimensional case for the system (1.2) they are

$$L=[f(y)+g(y)z]\frac{d}{dy}-az\frac{d}{dz}+\frac{b^2}{2}\frac{d^2}{dz^2}$$
$$\Lambda=b\frac{d}{dz} \tag{3.2}$$

3.1. Strong Explicit Schemes

The simplest method is the first-order explicit one

$$Y_{k+1}=Y_k+h[f+GZ]_k$$
$$Z_{k+1}=Z_k+\sum_{r=1}^{q}b_r\xi_{rk}h^{1/2}+hAZ_k \tag{3.3}$$

which can be proved by Theorem 1. Here ξ_{rk} are independent normally distributed $N(0, 1)$ random variables. The scheme (3.3) is the well-known Euler method for the system (1.1) with colored noise, which was presented by Fox *et al.*[22] who also proposed another first-order algorithm for differential equations with colored noise which was based on the exact solution of the Ornstein–Uhlenbeck process. Later, using this approach, Fox[12] derived the second-order scheme. Let us remark that using Theorem 1 one can prove that if the Gaussian process G_2 of Fox's scheme[12] is neglected, the strong order of this method would be still equal to 2. Nevertheless, the inclusion of G_2 does not lead to an increase of the scheme's order. It should be noted that as we know (for instance, refs. 10–12), nobody has rigorously proved the order of presented methods on the whole interval for a system with colored noise.

Using the Wagner–Platen expansion (2.4) and Theorem 1, we obtain the second-order explicit strong scheme for differential equations with colored noise (1.1):

$$Y_{k+1} = Y_k + [f + GZ]_k h + G_k \sum_{r=1}^{q} b_r I_{r0_k}$$
$$+ \frac{h^2}{2} [(f'_Y + (GZ)'_Y)(f + GZ) + GAZ]_k \tag{3.4}$$

$$Z_{k+1} = Z_k + \sum_{r=1}^{q} b_r I_{r_k} + AZ_k h + A \sum_{r=1}^{q} b_r I_{r0_k} + \frac{h^2}{2} A^2 Z_k$$

where $f_k = f(Y_k)$, $G_k = G(Y_k)$, f'_Y is a Jacobian matrix, $(GZ)'_Y = [G'_{y_1} Z\, G'_{y_2} Z \cdots G'_{y_l} Z]$ is an $l \times l$ matrix, $I_{r_k} = \xi_{rk} h^{1/2}$, and $I_{r0_k} = \frac{1}{2} h^{3/2}(\xi_{rk} + \eta_{rk}/\sqrt{3})$. Here ξ_{rk} and η_{rk} are independent random variables with standard normal distribution $N(0, 1)$. In the one-dimensional case the method (3.4) becomes

$$y_{k+1} = y_k + [f + gz]_k h + g_k b(\xi_k + \eta_k/\sqrt{3})\, h^{3/2}/2$$
$$+ h^2[(f' + g'z)(f + gz) - gaz]_k/2 \tag{3.5}$$
$$z_{k+1} = z_k + b\xi_k h^{1/2} - az_k h - ab(\xi_k + \eta_k/\sqrt{3})\, h^{3/2}/2 + h^2 a^2 z_k/2$$

In contrast to the Fox's second-order method,[12] in which he exactly simulated an Ornstein–Uhlenbeck process, our second-order explicit scheme (3.4) approximates Z with the same order as Y and so allows us to solve more general systems. Probably, the laboriousness of both methods is comparable.

If we add three terms to the second-order scheme (3.4), we obtain the 5/2-order strong algorithm

$$\tilde{Y}_{k+1} = Y_{k+1} + \sum_{r=1}^{q} [(Gb_r)'_Y (f + GZ)]_k I_{0r0_k} + \sum_{r=1}^{q} [f'_Y G b_r + GAb_r$$
$$+ A_r\{(GZ)'_Y f + (GZ)'_Y GZ\}]_k I_{r00_k}$$
$$+ \frac{h^3}{6} [L^2(f + GZ)]_k \tag{3.6}$$

$$\tilde{Z}_{k+1} = Z_{k+1} + \sum_{r=1}^{q} A^2 b_r I_{r00_k} + \frac{h^3}{6} A^3 Z_k$$

where Y_{k+1} and Z_{k+1} are taken from (3.4),

$$I_{0r0} = 2J_r - hI_{r0}, \qquad I_{r00} = hI_{r0} - J_r$$

$$J_r = \int_0^h v W_r(v)\, dv, \qquad I_{r0} = h^{3/2}(\xi_r + \eta_r/\sqrt{3})/2$$

$$J_r = h^{5/2}[\xi_r/3 + \eta_r/(4\sqrt{3}) + \zeta_r/(12\sqrt{5})]$$

ξ_r, η_r, and ζ_r are independent random variables with standard Gaussian distribution $N(0, 1)$ which are simulated at each step.

As mentioned above, for a general system (2.1) only the 1/2-order strong methods may be obtained with easily simulated random variables. The higher-order methods need numerical solution of a special system of SDE at each step for the simulation of the Ito integrals or some approximation of repeated Ito integrals in the case of the first-order scheme.[14] However, for the system with colored noise (1.1) efficient strong methods up to the 5/2 order are derived according to the special properties of the system (1.1). By the way, third-order schemes for the system (1.1) require calculation of repeated Ito integrals, and in the case of nonlinear functions f and G it is impossible to obtain an efficient third-order strong method with easily simulated random variables.

3.2. Runge–Kutta Strong Schemes

To reduce calculations of derivatives, we propose the explicit second-order strong Runge–Kutta scheme

$$\begin{aligned} Y_{k+1} &= Y_k + h\{[f+GZ]_k + [f+GZ]_{\bar{k}}\}/2 + \sum_{r=1}^{q} G_k b_r h^{3/2} \eta_{rk}/\sqrt{12} \\ Z_{k+1} &= Z_k + \sum_{r=1}^{q} b_r \xi_{rk} h^{1/2} + hA\{Z_k + Z_{\bar{k}}\}/2 + \sum_{r=1}^{q} A b_r h^{3/2} \eta_{rk}/\sqrt{12} \end{aligned} \tag{3.7}$$

where

$$f_{\bar{k}} = f(Y_{\bar{k}}), \qquad G_{\bar{k}} = G(Y_{\bar{k}}), \qquad Y_{\bar{k}} = Y_k + (f+GZ)_k h$$

$$Z_{\bar{k}} = Z_k + \sum_{r=1}^{q} b_r \xi_{rk} h^{1/2} + AZ_k h$$

This algorithm has been derived by the substitution of the expansions

$$\begin{aligned} ([f+GZ]_{\bar{k}} + [f+GZ]_k)/2 &= [f+GZ]_k + \sum_{r=1}^{q} G_k b_r \xi_{rk} h^{1/2}/2 \\ &\quad + L[f+GZ]_k h/2 + \rho_1 \\ (AZ_{\bar{k}} + AZ_k)/2 &= AZ_k + \sum_{r=1}^{q} A b_r \xi_{rk} h^{1/2}/2 + hA^2 Z_k/2 \\ \langle \rho_1 \rangle = O(h^2), &\qquad [\langle \rho_1^2 \rangle]^{1/2} = O(h^{3/2}) \end{aligned} \tag{3.8}$$

in the second-order scheme (3.4).

The Runge–Kutta scheme (3.7) does not include any derivatives; thanks to the special properties of the system (1.1) it is a "fully" Runge–Kutta algorithm.

The 5/2-order explicit method (3.6) may be simplified by the idea of attracting a subsidiary system of deterministic equations.[14] One has difficulties in the calculation of some terms of the method (3.6), for instance, $[f'_Y+(GZ)'_Y][f+GZ]$ and $L^2[f+GZ]$. Fortunately, these terms may be calculated by attracting the following subsidiary system:

$$dY/dt = F(Y, Z), \qquad dZ/dt = AZ \tag{3.9}$$

where $F(Y, Z) = f(Y) + G(Y)Z$. The system (3.9) may be solved by any deterministic third-order Runge–Kutta rule. Substituting a numerical solution of (3.9) in the scheme (3.6), we obtain the 5/2 "semi"-Runge–Kutta method:

$$\begin{aligned}
K_1 &= hF(Y_k, Z_k)\\
K_2 &= hF(Y_k + K_1/2, Z_k + hAZ_k/2)\\
K_3 &= hF(Y_k - K_1 + 2K_2, Z_k + hAZ_k + h^2A^2Z_k)
\end{aligned}$$

$$\begin{aligned}
Y_{k+1} &= Y_k + [K_1 + 4K_2 + K_3]/6 + G_k \sum_{r=1}^{q} b_r I_{r0_k}\\
&\quad + \sum_{r=1}^{q} [(Gb_r)'_Y (f + GZ)]_k I_{0r0_k}\\
&\quad + \sum_{r=1}^{q} [f'_Y Gb_r + GAb_r + A_r\{(GZ)'_Y f + (GZ)'_Y GZ\}]_k I_{r00_k}\\
&\quad + \frac{h^3}{6} \sum_{r=1}^{q} [(Gb_r)'_Y Gb_r]_k\\
Z_{k+1} &= Z_k + \sum_{r=1}^{q} b_r I_{r_k} + AZ_k h + A \sum_{r=1}^{q} b_r I_{r0_k} + \frac{h^2}{2} A^2 Z_k\\
&\quad + \sum_{r=1}^{q} A^2 b_r I_{r00_k} + \frac{h^3}{6} A^3 Z_k
\end{aligned} \tag{3.10}$$

where the needed Ito integrals are simulated in the same way as in (3.6). The correctness of the method (3.10) is proved by Theorem 1. Obviously, the algorithm (3.10) is simpler than (3.6). In contrast to the method (3.6), the scheme (3.10) does not include the second derivatives of functions f and G, and it also contains much fewer terms than (3.6). In the particular case of linear functions f and G the method (3.10) becomes a fully Runge–Kutta scheme.

3.3. Strong Implicit Schemes

For a stiff system it would be useful to have implicit methods.

A family of the first-order implicit methods (impicit Euler schemes) has the form

$$\begin{aligned} Y_{k+1} &= Y_k + \alpha h[f+GZ]_k + (1-\alpha)\,h[f+GZ]_{k+1} \\ Z_{k+1} &= Z_k + \sum_{r=1}^{q} b_r \xi_{rk} h^{1/2} + \alpha h A Z_k + (1-\alpha)\,h A Z_{k+1} \end{aligned} \tag{3.11}$$

where ξ_{rk} are independent normally distributed $N(0, 1)$ random variables, and $0 \leqslant \alpha \leqslant 1$.

We also present the two-parameter (α and β) family of second-order implicit schemes

$$\begin{aligned} Y_{k+1} &= Y_k + \alpha h[f+GZ]_k + (1-\alpha)\,h[f+GZ]_{k+1} \\ &\quad + h^{3/2} \sum_{r=1}^{q} G_k b_r((2\alpha-1)\,\xi_{rk}/2 \\ &\quad + \eta_{rk}/\sqrt{12}) + \beta(2\alpha-1)\,h^2 L[f+GZ]_k/2 \\ &\quad + (1-\beta)(2\alpha-1)\,h^2 L[f+GZ]_{k+1}/2 \\ Z_{k+1} &= Z_k + \sum_{r=1}^{q} b_r \xi_{rk} h^{1/2} + \alpha h A Z_k + (1-\alpha)\,h A Z_{k+1} \\ &\quad + h^{3/2} \sum_{r=1}^{q} A b_r((2\alpha-1)\,\xi_{rk}/2 \\ &\quad + \eta_{rk}/\sqrt{12}) + \beta(2\alpha-1)\,h^2 A^2 Z_k/2 \\ &\quad + (1-\beta)(2\alpha-1)\,h^2 A^2 Z_{k+1}/2 \end{aligned} \tag{3.12}$$

where ξ_{rk} and η_{rk} are independent, normally distributed $N(0, 1)$ random variables, and $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant 1$. The family (3.12) is derived by representing the terms $[f+GZ]_k$ and AZ_k of (3.4) in the form

$$\begin{aligned} [f+GZ]_k &= \alpha[f+GZ]_k + (1-\alpha)\{[f+GZ]_{k+1} \\ &\quad - \sum_{r=1}^{q} G_k b_r \xi_{rk} h^{1/2} - hL[f+GZ]_k + \rho_1\} \\ AZ_k &= \alpha A Z_k + (1-\alpha)\left(AZ_{k+1} - \sum_{r=1}^{q} A b_r \xi_{rk} h^{1/2} - hA^2 Z_k + \rho_2\right) \end{aligned} \tag{3.13}$$

$$\langle \rho_i \rangle = O(h^2), \qquad [\langle \rho_i^2 \rangle]^{1/2} = O(h^{3/2}), \qquad i = 1, 2$$

and with the expressions

$$\begin{aligned} L[f+GZ]_k &= \beta L[f+GZ]_k + (1-\beta)\, L[f+GZ]_{k+1} + \rho_3 \\ A^2 Z_k &= \beta A^2 Z_k + (1-\beta)\, A^2 Z_{k+1} + \rho_4 \\ \langle \rho_i \rangle = O(h), & \qquad [\langle \rho_i^2 \rangle]^{1/2} = O(h^{1/2}), \qquad i = 3, 4 \end{aligned} \tag{3.14}$$

The approach to derivation of such implicit methods as (3.12) was first presented in ref. 14, where 3/2-order implicit schemes for a general system with additive noises were obtained. In spite of the general case, the implicit methods (3.12) are of the second order thanks to the special properties of the system (1.1) with colored noise. If we choose $\alpha = 1/2$ in (3.12), we obtain the simplest scheme of the family (3.12), which is called the trapezoidal method:

$$\begin{aligned} Y_{k+1} &= Y_k + h\{[f+GZ]_k + [f+GZ]_{k+1}\}/2 \\ &\quad + h^{3/2} \sum_{r=1}^{q} G_k b_r \eta_{rk}/\sqrt{12} \\ Z_{k+1} &= Z_k + \sum_{r=1}^{q} b_r \xi_{rk} h^{1/2} + hA[Z_k + Z_{k+1}]/2 \\ &\quad + h^{3/2} \sum_{r=1}^{q} A b_r \eta_{rk}/\sqrt{12} \end{aligned} \tag{3.15}$$

The random variables ξ_{rk} and η_{rk} are independent and normally distributed $N(0, 1)$. The Appendix gives a proof of the trapezoidal method (3.15).

It is also possible to construct an implicit Runge–Kutta method.

4. WEAK SCHEMES FOR SYSTEMS WITH COLORED NOISE

Several weak methods for the differential equations with colored noise (1.1) are presented in this section. They are second- and the third-order explicit schemes, and implicit and Runge–Kutta algorithms. The methods may be derived using the corresponding strong schemes of Section 3 and the theorem on the relation between the properties of the one-step weak approximation and the order of the method.[14,15,19,20]

4.1. Weak Explicit Schemes

The first-order weak method coincides with the Euler scheme.

The second-order explicit weak method is

$$
\begin{aligned}
Y_{k+1} &= Y_k + [f+GZ]_k h + h^{3/2} G_k \sum_{r=1}^{q} b_r \xi_{rk}/2 \\
&\quad + h^2[(f'_Y + (GZ)'_Y)(f+GZ) + GAZ]_k/2 \qquad (4.1) \\
Z_{k+1} &= Z_k + h^{1/2} \sum_{r=1}^{q} b_r \xi_{rk} + AZ_k h + h^{3/2} A \sum_{r=1}^{q} b_r \xi_{rk}/2 + h^2 A^2 Z_k/2
\end{aligned}
$$

where ξ_{rk} are independent random variables with standard normal distribution $N(0, 1)$ or distributed according to the laws $P(\xi = 0) = 2/3$, $P(\xi = -\sqrt{3}) = P(\xi = \sqrt{3}) = 1/6$, where P is the probability of random variable ξ. Other symbols have the same sense as in Section 3.

The third-order weak method is

$$
\begin{aligned}
Y_{k+1} &= Y_k + h[f+GZ]_k + h^{3/2} G_k \sum_{r=1}^{q} b_r(\xi_{rk}/2 + v_{rk}) \\
&\quad + h^2[(f'_Y + (GZ)'_Y)(f+GZ) + GAZ]_k/2 \\
&\quad + h^{5/2} \sum_{r=1}^{q} [(Gb_r)'_Y (f+GZ)]_k (\xi_{rk}/6 - v_{rk}) \\
&\quad + h^{5/2} \sum_{r=1}^{q} [f'_Y G b_r + GAb_r + A_r\{(GZ)'_Y f + (GZ)'_Y GZ\}]_k \qquad (4.2) \\
&\quad \times (\xi_{rk}/6 + v_{rk}) + h^3 L^2 [f+GZ]_k/6 \\
Z_{k+1} &= Z_k + h^{1/2} \sum_{r=1}^{q} b_r \xi_{rk} + AZ_k h + h^{3/2} A \\
&\quad \times \sum_{r=1}^{q} b_r[\xi_{rk}/2 + v_{rk}] + h^2 AZ_k/2 \\
&\quad + h^{5/2} A^2 \sum_{r=1}^{q} b_r(\xi_{rk}/6 + v_{rk}) + h^3 A^3 Z_k/6
\end{aligned}
$$

The independent random variables ξ_{rk} and v_{rk} of (4.2) may be simulated as $N(0, 1)$ and $N(0, 1/\sqrt{12})$, respectively. Here $N(0, \Delta)$ is a Gaussian distribution with zero mean value and standard deviation Δ. One can obtain the random variables ξ_r and v_r in another manner by the laws

$$
\begin{aligned}
&P(v = -1/\sqrt{12}) = P(v = 1/\sqrt{12}) = 1/2, \\
&P(\xi = -1) = P(\xi = 1) = 3/10 \\
&P(\xi = 0) = 1/3, \qquad P(\xi = -\sqrt{6}) = P(\xi = \sqrt{6}) = 1/30
\end{aligned}
$$

4.2. The Runge–Kutta Weak Schemes

The second-order explicit Runge–Kutta method takes the form

$$\begin{aligned} Y_{k+1} &= Y_k + h\{[f+GZ]_k + [f+GZ]_{\bar{k}}\}/2 \\ Z_{k+1} &= Z_k + h^{1/2}\sum_{r=1}^{q} b_r\xi_{rk} + hA[Z_{\bar{k}} + Z_k]/2 \end{aligned} \tag{4.3}$$

where

$$f_{\bar{k}} = f(Y_{\bar{k}}), \qquad G_{\bar{k}} = G(Y_{\bar{k}}), \qquad Y_{\bar{k}} = Y_k + (f+GZ)_k h$$

$$Z_{\bar{k}} = Z_k + h^{1/2}\sum_{r=1}^{q} b_r\xi_{rk} + AZ_k h$$

The random variable ξ_{rk} are the same as in the scheme (4.1).

Similar to the 5/2 Runge–Kutta strong method (3.10), the third-order Runge–Kutta weak method may be obtained with random variables ξ_{rk} and ν_{rk} as in the third-order explicit weak method (4.2).

4.3. Weak Implicit Methods

The first-order implicit weak methods coincide with the Euler strong schemes (3.11), but independent random variables ξ_{rk} may be simulated as $P(\xi = 1) = P(\xi = -1) = 1/2$.

The two-parameter family of second-order implicit weak schemes has the form

$$\begin{aligned} Y_{k+1} &= Y_k + \alpha h[f+GZ]_k + (1-\alpha)h[f+GZ]_{k+1} \\ &\quad + h^{3/2}\sum_{r=1}^{q} G_k b_r(2\alpha-1)\xi_{rk}/2 \\ &\quad + \beta(2\alpha-1)h^2L[f+GZ]_k/2 + (1-\beta)(2\alpha-1)h^2L[f+GZ]_{k+1}/2 \\ Z_{k+1} &= Z_k + \sum_{r=1}^{q} b_r\xi_{rk}h^{1/2} + \alpha hAZ_k + (1-\alpha)hAZ_{k+1} \\ &\quad + h^{3/2}\sum_{r=1}^{q} Ab_r(2\alpha-1)\xi_{rk}/2 \\ &\quad + \beta(2\alpha-1)h^2A^2Z_k/2 + (1-\beta)(2\alpha-1)h^2A^2Z_{k+1}/2 \end{aligned} \tag{4.4}$$

The random variables ξ_{rk} are the same as in the scheme (4.1), and $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant 1$.

If parameter α in (4.4) is equal to 1/2, we obtain the trapezoidal weak method

$$Y_{k+1} = Y_k + h\{[f+GZ]_k + [f+GZ]_{k+1}\}/2$$
$$Z_{k+1} = Z_k + h^{1/2}\sum_{r=1}^{q} b_r\xi_{rk} + hA[Z_{k+1}+Z_k]/2 \tag{4.5}$$

which is the simplest one among the family (4.4).

It is also possible to derive weak implicit Runge–Kutta schemes.

5. NUMERICAL TESTS

We test the presented numerical methods on the well-known Kubo oscillator with random frequency[23,24]

$$dy/dt = iy(\omega_0 + z(t)) \tag{5.1}$$

where y is a complex variable and $z(t)$ is the Ornstein–Uhlenbeck process as defined by (1.2)–(1.3). The Kubo oscillator has seen application in the theory of nuclear magnetic resonance.[23] As mentioned in refs. 10 and 12, the complex equation (5.1) is an excellent choice for testing the quality of algorithms. It has analytic solution, so that one can compare the simulations with explicit formulas,[24] for instance,

$$y(t) = y(0)\exp[i\omega_0 t + i\int_0^t z(t')\,dt'] \tag{5.2a}$$

$$yy^* \equiv 1 \tag{5.2b}$$

$$\langle y(t)\rangle = \langle y(0)\rangle \exp\{i\omega_0 t - b^2\{t-(1-\exp[-at])/a\}(2a^2)\} \tag{5.2c}$$

$$\begin{aligned}\langle u^2(t)\rangle = \{&(\langle u(0)^2\rangle - \langle v(0)^2\rangle)\cos(2\omega_0 t)\\ &-\langle v(0)\,u(0)\rangle \sin(2\omega_0 t)\}\\ &\times\exp\{-b^2[t-(1-\exp[-at])/a]/a^2\}/2 + 1/2\end{aligned} \tag{5.2d}$$

where $u = \operatorname{Re} y$ and $v = \operatorname{Im} y$. By the way, it is impossible to check all features of an algorithm by simulation of the Kubo oscillator which is linear with respect to y and symmetric.

For the SDE (5.1) the vectors Y, f, b_r, etc., are given by

$$Y = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad f = \begin{pmatrix} -\omega_0 & v \\ \omega_0 & u \end{pmatrix}, \qquad G = \begin{pmatrix} -v \\ u \end{pmatrix},$$
$$q = 1, \qquad Z = z, \qquad b_1 = b, \qquad A = -a \tag{5.3}$$
$$f'_Y = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}, \qquad (GZ)'_Y = \begin{pmatrix} 0 & -z \\ z & 0 \end{pmatrix}, \qquad \text{etc.}$$

5.1. Strong Schemes Test

We test our strong methods by comparing equality (5.2b) with a simulated oscillator trajectory (see Fig. 1).

According to Section 2, the strong algorithms for the SDE (5.1) are written in the following form:

Second-order strong scheme:

$$\begin{aligned}
&\text{coef1} = 1 - h^2(\omega_0 + z_k)^2/2 \\
&\text{coef2} = (\omega_0 + z_k)h + bh^{3/2}[\xi_k + \eta_k/\sqrt{3}]/2 - h^2 a z_k/2 \\
&u_{k+1} = u_k\,\text{coef1} - v_k\,\text{coef2} \qquad (5.4)\\
&v_{k+1} = v_k\,\text{coef1} + u_k\,\text{coef2} \\
&z_{k+1} = z_k + h^{1/2} b\xi_k - a z_k h - abh^{3/2}[\xi_k + \eta_k/\sqrt{3}]/2 + h^2 a^2 z_k/2
\end{aligned}$$

ξ_k and η_k are independent random variables with standard normal distribution.

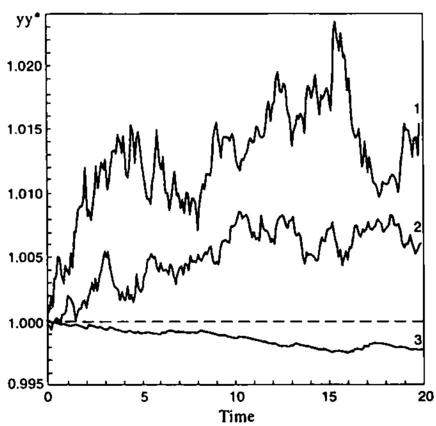

Fig. 1. Test of strong methods. Time dependence of Kubo oscillator radius yy^* for $\omega_0 = 1$, $a = 1$, $b = 0.33$, $u(0) = 0$, $v(0) = 1$, $h = 0.1$. One stochastic realization is simulated by (1) the second-order explicit scheme (5.4), (2) the trapezoidal method (5.6), and (3) the 5/2-order explicit scheme (5.5). Dashed line is the exact solution (5.2b).

5/2-order strong scheme:

$$\begin{aligned}
\mathrm{coef1} &= 1 - h^2(\omega_0 + z_k)^2/2 - bh^{5/2}(\omega_0 + z_k)(\xi_k + \eta_k/\sqrt{3})/2 \\
&\quad + h^3\{3az_k(\omega_0 + z_k) - b^2\}/6 \\
\mathrm{coef2} &= (\omega_0 + z_k)h + bh^{3/2}[\xi_k + \eta_k/\sqrt{3}]/2 - h^2 a z_k/2 \\
&\quad - abh^{5/2}\{\xi_k/6 + \eta_k/(4\sqrt{3}) - \zeta_k/(12\sqrt{5})\} \\
&\quad + h^3\{a^2 z_k - (\omega_0 + z_k)^3\}/6 \qquad (5.5) \\
u_{k+1} &= u_k\,\mathrm{coef1} - v_k\,\mathrm{coef2} \\
v_{k+1} &= v_k\,\mathrm{coef1} + u_k\,\mathrm{coef2} \\
z_{k+1} &= z_k + h^{1/2} b\xi_k - az_k h - abh^{3/2}[\xi_k + \eta_k/\sqrt{3}]/2 + h^2 a^2 z_k/2 \\
&\quad + a^2 bh^{5/2}\{\xi_k/6 + \eta_k/(4\sqrt{3}) - \zeta_k/(12\sqrt{5})\} - h^3 a^3 z_k/6
\end{aligned}$$

ξ_k, η_k and ζ_k are independent random variables with standard normal distribution.

Trapezoidal strong scheme:

$$\begin{aligned}
\gamma &= 1/(1 + ha/2) \\
\mathrm{coef1} &= h(\omega_0 + z_k)/2 + b\eta_k h^{3/2}/\sqrt{12} \\
z_{k+1} &= \gamma[z_k(1 - ha/2) + b\xi_k h^{1/2} - abh^{3/2}\eta_k/\sqrt{12} \\
\mathrm{coef2} &= h(\omega_0 + z_{k+1})/2 \\
\varkappa &= 1/(1 + \mathrm{coef2}^2) \qquad (5.6) \\
\mathrm{coef11} &= 1 - \mathrm{coef2}\,\mathrm{coef1} \\
\mathrm{coef22} &= \mathrm{coef1} + \mathrm{coef2} \\
u_{k+1} &= \varkappa(u_k\,\mathrm{coef11} - v_k\,\mathrm{coef22}) \\
v_{k+1} &= \varkappa(v_k\,\mathrm{coef11} + u_k\,\mathrm{coef22})
\end{aligned}$$

ξ_k and η_k are independent random variables with standard normal distribution.

To generate Gaussian random numbers, we use procedure GASDEV from ref. 25. Figure 1 demonstrates the time dependence of the Kubo oscillator radius yy^* for one stochastic trajectory simulated by strong schemes (5.4)–(5.6). It confirms the correctness and orders of the methods.

5.2. Weak Schemes Test

We test our weak methods by comparing simulations of the SDE (5.1) with explicit formulas (5.2b), (5.2c), and (5.2d) (see tables and Fig. 2).

According to Section 4, the weak algorithms for the SDE (5.1) are rewritten in the following form:

Second-order weak scheme:

$$\begin{aligned}
\mathrm{coef1} &= 1 - h^2(\omega_0 + z_k)^2/2 \\
\mathrm{coef2} &= (\omega_0 + z_k)h + bh^{3/2}\xi_k/2 - h^2 a z_k/2 \\
u_{k+1} &= u_k\,\mathrm{coef1} - v_k\,\mathrm{coef2} \\
v_{k+1} &= v_k\,\mathrm{coef1} + u_k\,\mathrm{coef2} \\
z_{k+1} &= z_k + h^{1/2}b\xi_k - az_k h - abh^{3/2}\xi_k/2 + h^2a^2z_k/2
\end{aligned} \tag{5.7}$$

ξ_k are independent random variables with distribution according to the laws $P(\xi = 0) = 2/3$, $P(\xi = -\sqrt{3}) = P(\xi = \sqrt{3}) = 1/6$.

Third-order weak scheme:

$$\begin{aligned}
\mathrm{coef1} = {} & 1 - h^2(\omega_0 + z_k)^2/2 - h^{5/2}b(\omega_0 + z)[\xi_k/2 + \nu_k] \\
& + h^3\{3az_k(\omega_0 + z_k) - b^2\}/6 \\
\mathrm{coef2} = {} & (\omega_0 + z)h + h^{3/2}b(\xi_k/2 + \nu_k) - h^2az_k/2 - h^{5/2}ab(\xi_k/6 + \nu_k) \\
& - h^3\{(\omega_0 + z_k)^3 - a^2z_k\}/6
\end{aligned}$$

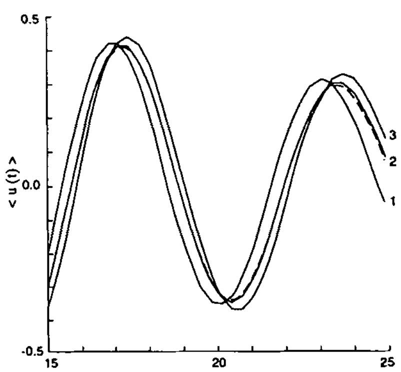

Fig. 2. Test of weak methods. Time dependence of mean value of $u = \mathrm{Re}\, y$ for $\omega_0 = 1$, $a = 1$, $b = 0.33$, $u(0) = 0$, $v(0) = 1$, $h = 0.3$ with averages over 10,000 realizations. (1) The second-order explicit scheme (5.7), (2) the third-order explicit scheme (5.8), (3) the trapezoidal method (5.9). Dashed line is the exact solution (5.2c). The Monte Carlo error is approximately equal to 10^{-2} and less than the method errors.

$$u_{k+1} = \text{coef1}\ u_k - \text{coef2}\ v_k$$
$$v_{k+1} = \text{coef1}\ v_k + \text{coef2}\ u_k$$
$$z_{k+1} = z_k + h^{1/2} b\xi_k - az_k h - h^{3/2} ab[\xi_k/2 + v_k] + h^2 a^2 z_k/2$$
$$+ a^2 b h^{5/2}\{\xi_k/6 + v_k\} - h^3 a^3 z_k/6 \tag{5.8}$$

ξ_k and v_k are independent variables with distributions according to the laws

$$P(\xi = 0) = 1/3, \qquad P(\xi = -\sqrt{6}) = P(\xi = \sqrt{6}) = 1/30$$
$$P(\xi = -1) = P(\xi = 1) = 3/10,$$
$$P(v = -1/\sqrt{12}) = P(v = 1/\sqrt{12}) = 1/2$$

Trapezoidal weak scheme:

$$\begin{aligned} \gamma &= 1/(1 + ha/2) \\ \text{coef1} &= h(\omega_0 + z_k)/2 \\ z_{k+1} &= \gamma[z_k(1 - ha/2) + b\xi_k h^{1/2}] \\ \text{coef2} &= h(\omega_0 + z_{k+1})/2 \\ \varkappa &= 1/(1 + \text{coef2}^2) \\ \text{coef11} &= 1 - \text{coef2 coef1} \\ \text{coef22} &= \text{coef1} + \text{coef2} \\ u_{k+1} &= \varkappa(u_k\, \text{coef11} - v_k\, \text{coef22}) \\ v_{k+1} &= \varkappa(v_k\, \text{coef11} + u_k\, \text{coef22}) \end{aligned} \tag{5.9}$$

The random variables ξ_k are the same as in (5.7).

To generate uniform random numbes we use the procedure RAN1 from ref. 25. The initial value $z(0)$ is simulated as a Gaussian random number with zero mean and variance $b^2/(2a)$.

In Tables I and II we present simulations of mean values of $u = \text{Re}\, y$ for various steps h and various numbers of stochastic realizations N. The values in Tables I and II approximate $\langle \bar{u}(t) \rangle$ [$\bar{u}(t)$ is the numerical solution of (5.1) given by weak schemes (5.7)–(5.9)] calculated as

$$\langle \bar{u}(t) \rangle \approx \frac{1}{N} \sum_{m=1}^{N} \bar{u}^{(m)}(t)$$
$$\pm \frac{2}{\sqrt{N}} \left[\frac{1}{N} \sum_{m=1}^{N} (\bar{u}^{(m)})^2(t) - \left(\frac{1}{N} \sum_{m=1}^{N} \bar{u}^{(m)}(t) \right)^2 \right]^{1/2} \tag{5.10}$$

Table I. Test of Weak Methods: Computational Results for Mean Value of $u = \mathrm{Re}\, y$ for $t = 10$[a]

h	N	Explicit second order	Explicit third order	Trapezoidal method	$2[(\langle u^2(10)\rangle - \langle u(10)\rangle^2)/N]^{1/2}$
		$\frac{1}{N}\sum_{i=1}^{N} \bar{u}^{(i)}(10) \pm 2\left[\left(\frac{1}{N}\sum_{i=1}^{N} [\bar{u}^{(i)}(10)]^2 - \left[\frac{1}{N}\sum_{i=1}^{N} \bar{u}^{(i)}(10)\right]^2\right)\Big/N\right]^{1/2}$			
0.4	5,000	0.4748 ± 0.0173	0.3116 ± 0.0165	0.2609 ± 0.0174	0.0158
0.2	5,000	0.3649 ± 0.0167	0.3348 ± 0.0169	0.3121 ± 0.0170	0.0158
0.1	5,000	0.3541 ± 0.0166	0.3461 ± 0.0167	0.3415 ± 0.0166	0.0158
0.05	5,000	0.3490 ± 0.0166	0.3456 ± 0.0166	0.3458 ± 0.0166	0.0158
0.2	100,000	0.3719 ± 0.0037	0.3318 ± 0.0037	0.3198 ± 0.0038	0.0035
0.1	100,000	0.3499 ± 0.0037	0.3386 ± 0.0037	0.3372 ± 0.0037	0.0035

[a] For $\omega_0 = 1$, $a = 1$, $b = 0.33$, $u(0) = 0$, $v(0) = 1$, and various steps h with averages over N realizations. The exact solution is $\langle u(10)\rangle = 0.3333$.

i.e., $\langle \bar{u}(t)\rangle$ belongs to the interval defined by (5.10) with probability 0.95 under the assumption that the sampling variance is sufficiently close to the variance of $\bar{u}(t)$. It is obvious that the values in the tables include first the error of the method and second the Monte Carlo error. If $N \to \infty$, then the Monte Carlo error diminishes and the difference between the value

Table II. Test of Weak Methods: Computational Results for Mean Value of $u = \mathrm{Re}\, y$ for $t = 20$[a]

h	N	Explicit second order	Explicit third order	Trapezoidal method	$2[(\langle u^2(20)\rangle - \langle u(20)\rangle^2)/N]^{1/2}$
		$\frac{1}{N}\sum_{i=1}^{N} \bar{u}^{(i)}(20) \pm 2\left[\left(\frac{1}{N}\sum_{i=1}^{N} [\bar{u}^{(i)}(20)]^2 - \left[\frac{1}{N}\sum_{i=1}^{N} \bar{u}^{(i)}(20)\right]^2\right)\Big/N\right]^{1/2}$			
0.2	5,000	-0.3589 ± 0.0179	-0.3435 ± 0.0175	-0.3422 ± 0.0177	0.0187
0.1	5,000	-0.3604 ± 0.0175	-0.3499 ± 0.0174	-0.3566 ± 0.0175	0.0187
0.05	5,000	-0.3590 ± 0.0176	-0.3472 ± 0.0175	-0.3584 ± 0.0176	0.0187
0.2	100,000	-0.3588 ± 0.0040	-0.3379 ± 0.0039	-0.3415 ± 0.0040	0.0042

[a] For the same values as in Table I. The exact solution is $\langle u(20)\rangle = -0.3244$.

Table III. Test of Weak Methods. The Kubo Oscillator Radius Simulations $\bar{y}\bar{y}^*$ for $t = 10^a$

		$\frac{1}{N}\sum_{i=1}^{N} \bar{y}\bar{y}^{*(i)}(10) \pm 2\left[\left(\frac{1}{N}\sum_{i=1}^{N}[\bar{y}\bar{y}^{*(i)}(10)]^2 - \left[\frac{1}{N}\sum_{i=1}^{N}\bar{y}\bar{y}^{*(i)}(10)\right]^2\right)\Big/N\right]^{1/2}$			
h	N	Explicit second order	Explicit third order	Trapezoidal method	Fox[12]
0.4	5,000	1.1832 $\pm$0.0032	0.935 $\pm$0.001	1.00036 $\pm$0.00072	
0.2	5,000	1.0163 $\pm$0.0049	0.99102 $\pm$0.00017	0.99998 $\pm$0.00019	
0.1	5,000	1.00063$\pm$0.00010	0.99885 $\pm$0.00002	1.00004 $\pm$0.00005	1.00321
0.05	5,000	0.99974$\pm$0.00002	0.999855$\pm$0.000003	1.000003$\pm$0.000012	
0.2	100,000	1.01613$\pm$0.00011	0.99101 $\pm$0.00004	1.00012 $\pm$0.00004	
0.1	100,000	1.00069$\pm$0.00002	0.998847$\pm$0.000005	1.000006$\pm$0.000011	

[a] For $\omega_0 = 1$, $a = 1$, $b = 0.33$ (Fox[12] $b^2 = 0.1$), $u(0) = 0$, $v(0) = 1$, and various steps h with averages over N realizations. The exact solution is $yy^* \equiv 1$.

in the table and the exact solution tends to the method error. Analyzing Tables I, II, and Fig. 2, one can compare the presented weak schemes and prove their correctness.

Tables III and IV give averaged values for the Kubo oscillator radius $\bar{y}\bar{y}^*$. These results also prove the correctness of the algorithms. One can see that in some cases (for instance, Table III, $h = 0.4$, $N = 5000$) the trapezoidal method gives better results for the radius yy^* than the second- and the third-order schemes. This is explained by the fact that in contrast to the explicit schemes (5.7), (5.8), the trapezoidal method (5.9) exactly gives the expression $\langle y_k y_k^* \rangle = 1$, and the method error for this moment is equal to zero. This is caused by the specific properties of the SDE (5.1).

Table IV. Test of Weak Methods. The Kubo Oscillator Radius Simulations for $t = 20^a$

		$\frac{1}{N}\sum_{i=1}^{N} \bar{y}\bar{y}^{*(i)}(20) \pm 2\left[\left(\frac{1}{N}\sum_{i=1}^{N}[\bar{y}\bar{y}^{*(i)}(20)]^2 - \left[\frac{1}{N}\sum_{i=1}^{N}\bar{y}\bar{y}^{*(i)}(20)\right]^2\right)\Big/N\right]^{1/2}$			
h	N	Explicit second order	Explicit third order	Trapezoidal method	Fox[12]
0.2	5,000	1.03257 $\pm$0.00058	0.9821 $\pm$0.0002	1.00012 $\pm$0.0019	
0.1	5,000	1.00134 $\pm$0.00011	0.997696$\pm$0.000033	1.000023$\pm$0.000046	1.00990
0.05	5,000	0.999498$\pm$0.000025	0.999710$\pm$0.000005	0.999999$\pm$0.000012	
0.2	100,000	1.03273 $\pm$0.00013	0.98209 $\pm$0.00005	1.000037$\pm$0.000042	

[a] Same values as in Table III.

6. CONCLUSIONS

Differential equations with colored noise have been investigated by van Kampen,[1] Horthemke and Lefever,[6] Lindenberg,[8,9] Fox,[10,12,22] Risken,[24] and others.

The present paper develops a constructive approach to numerical integration of differential equations with colored noise. We discuss two types of approximations of stochastic differential equations, strong and weak approximations. We present several strong schemes, including the 5/2-order explicit method and Runge–Kutta and implicit algorithms. We also obtain various weak methods for the system with colored noise, namely up to third-order explicit schemes and Runge–Kutta and implicit methods. All of the presented algorithms are efficient as to simulation of the used random variables. For instance, in the case of the 5/2 strong order method for the system (1.1) with q colored noises one must simulate $3q$ Gaussian independent variables at each integration step and in the case of the third-weak-order scheme at each integration step one must generate only $2q$ independent random variables with simple distributions. We succeeded in constructing efficient high-order methods because of the special properties of the system with colored noise. The higher-order methods cannot be efficient, even for this system.

By the approach developed in this paper one can derive other weak and strong algorithms for the system with colored noise. It is also possible to consider nonautonomous systems with colored noise and to construct appropriate numerical methods.

APPENDIX. DERIVATION OF THE STRONG TRAPEZOIDAL METHOD

Derivation of the strong trapezoidal method is based on the proof of a family of strong implicit schemes which was presented in ref. 14.

Using the Ito formula, we obtain

$$\begin{aligned}
&[f+GZ]_k/2=\left([f+GZ]_{k+1}-\sum_{r=1}^{q} G_k b_r I_{r_k}-hL[f+GZ]_k+\rho_1\right)\Big/2 \\
&\qquad AZ_k/2=\left(AZ_{k+1}-\sum_{r=1}^{q} Ab_r I_{r_k}-hA^2Z_k+\rho_2\right)\Big/2
\end{aligned} \tag{A.1}$$

where

$$
\begin{aligned}
\rho_1 = &-\sum_{r=1}^{q}\int_{t_k}^{t_{k+1}}\int_{t_k}^{\theta}[G(Y(\theta_1))\,b_r]'_Y\,[f(Y(\theta_1))+G(Y(\theta_1))\,Z(\theta_1)]\,d\theta_1\,dW_r \\
&-\sum_{r=1}^{q}\int_{t_k}^{t_{k+1}}\int_{t_k}^{\theta}\Lambda_r L[f+GZ]\,dW_r(\theta_1)\,d\theta \\
&-\int_{t_k}^{t_{k+1}}\int_{t_k}^{\theta}L^2[f+GZ]\,d\theta_1\,d\theta
\end{aligned}
\tag{A.2}
$$

$$
\rho_2 = -\sum_{r=1}^{q}A^2 b_r^2 I_{0r_k} - A^3\int_{t_k}^{t_{k+1}}\int_{t_k}^{\theta}Z(\theta_1)\,d\theta_1\,d\theta
$$

Using the properties of stochastic integrals, one can obtain

$$
\langle \rho_i\rangle = O(h^2), \qquad [\langle \rho_i^2\rangle]^{1/2} = O(h^{3/2}), \qquad i=1,2 \tag{A.3}
$$

Retaining the terms $[f+GZ]_k/2$ and $AZ_k/2$ in the right-hand sides of the second-order explicit method (3.4) and subsituting (A.1) in (3.4), we obtain the approximation

$$
\begin{aligned}
Y_{k+1} &= Y_k + h\{[f+GZ]_k + [f+GZ]_{k+1}\}/2 \\
&\quad + \sum_{r=1}^{q} G_k b_r[I_{r0_k} - I_{r_k}h/2] + h\rho_1 \\
Z_{k+1} &= Z_k + \sum_{r=1}^{q} b_r I_{r_k} + hA[Z_k + Z_{k+1}]/2 \\
&\quad + A\sum_{r=1}^{q} b_r[I_{r0_k} - I_{r_k}h/2] + h\rho_2
\end{aligned}
\tag{A.4}
$$

From (A.3) it follows that

$$
\langle h\rho_i\rangle = O(h^3), \qquad [\langle h^2\rho_i^2\rangle]^{1/2} = O(h^{5/2}), \qquad i=1,2 \tag{A.5}
$$

Then by Theorem 1 (see Section 2), using (A.5) and the fact that the scheme (3.4) is a second-strong-order method, one can prove that the strong order of the method, based on the approximation (A.4), is equal to 2 on the whole interval.

To realize the trapezoidal method, which is obtained from (A.4) by droping remainders, one must be able to calculate Ito integrals I_r and I_{r0}, which are Gaussian random variables. It is easy to obtain that

$$
\begin{aligned}
&\langle I_r\rangle = \langle I_{r0}\rangle = 0, \qquad \langle I_r^2\rangle = h, \\
&\langle I_{r0}^2\rangle = h^3/3, \qquad \langle I_r I_{r0}\rangle = h^2/2
\end{aligned}
\tag{A.6}
$$

Using two independent standard Gaussian random variables ξ_r and η_r, we can simulate these integrals in the following way:

$$I_r = h^{1/2}\xi_r, \qquad I_{r0} = h^{3/2}(\xi_r + \eta_r/\sqrt{3})/2 \tag{A.7}$$

Substituting (A.7) in the approximation (A.4) and dropping remainders, we obtain the trapezoidal method (3.15), which has the second strong order on the whole interval.

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